

Optimal quantum parameter estimation of two interacting qubits under decoherence

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We investigate the parameter estimation problem in a two-qubit system, in which each qubit is independently interacting with its Markovian environment. We study in detail the sensitivity of the estimation on the decoherence rate γ and the two-qubit interaction strength v . In particular, the dynamics of quantum Fisher information are employed as a measure to quantify the precision of the estimations. We find that the quantum Fisher information with respect to the decoherence rate scales like $t^2/[\exp(\gamma t) - 1]$ and $\exp(-2\gamma t)$ in the unitary limit and the completely decoherent limit, respectively. When we estimate the interaction strength v , the quantum Fisher information shows oscillation behavior in term of time. In addition, our results provide further evidence that the entanglement of the input state may not enhance quantum metrology.

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I. INTRODUCTION

Each quantum system is in contact with its environment. Understanding the dynamics of physical quantities under the effect of its decoherence environment has attracted much more interest. On a foundational level, time is a basically physical quantity. So the dynamical evolution is an important property of system, which makes the finite-time quantum quantities interesting in their own right [1]. An well-known example is that the entanglement dynamics of two qubits exhibits entanglement sudden death [2] in the decoherent environments.

The environmental noise presented in the physical system often determines the performance of quantum property. Therefore, it is important to develop methods to estimate the level of noise as precisely as possible. Determining the environmental parameters affecting a quantum system is the first step to develop means to control its spoiling effects. Except quantum process tomography [3], quantum channel estimation [4] is an efficient way to identify an unknown noise by estimating noise parameters. A general description of quantum channel estimation is as follows: for a prepared state ρ as an input of quantum channel Υ_θ , the channel parameters may be estimated efficiently by performing some quantum state measurements on the output state $\Upsilon_\theta(\rho)$. Thus, one should seek an optimal input state ρ and/or an optimal measurement on the output state $\Upsilon_\theta(\rho)$. Very recently, based on optical setup, the experimental realization of optimal estimation is reported for a Pauli noisy channel [5].

Fisher information [6] is a key quantity in classical estimation theory. Extending to quantum regime, quantum Fisher information (QFI) is also very important in quantum estimation theory and quantum information theory. QFI characterizes the sensitivity of a state with respect to changes on a parameter [8]. It is also related to Cramér-Rao inequality [7], which determines the bound of the optimal measurement. In the field of quantum estimation, the aim is to determine the value of the unknown

parameter labeling the quantum system, and the primary goal is to enhance the precision of resolution. Moreover, QFI is also closely related with other quantity, especially entanglement [9].

Previously, Huelga *et. al.* have discussed the optimal precision of frequency measurements in the presence of decoherence [10]. In a recent paper, a general theory for the quantum metrology of noisy systems has been proposed by Escher and co-workers [11]. More recently, the dynamics of QFI under decoherence excites wide interest. It has been shown that the evolution of QFI under decoherence for the N -qubit GHZ state shows the decay and sudden change [12]. For a spin- j system surrounded by a quantum critical Ising chain, the QFI decays almost monotonously when the environment reaches the critical point [13]. The dynamics of QFI for a qubit subject to a non-Markovian environment shows revival and retardation loss [14].

The main aim of this paper is to examine the problem of parameter estimation for two initially entangled qubits under decoherence. In particular, we focus on the dynamical evolution of quantum Fisher information. Quantum Fisher information shows the revival-like behavior in the Markovian environment when the estimated parameter is their interacting strength. This generalizes the result in Ref. [14]. We also show that quantum Fisher information can not be enhanced by the entanglement of initially mixed state. This extends the general result that the entangled states can improve the precision of parameter estimation [15, 16] from the view point of QFI dynamics.

II. QUANTUM FISHER INFORMATION

The classical Fisher information is defined as

$$F_{\theta, c} = \sum_i p_i(\theta) \left[\frac{\partial}{\partial \theta} \ln p_i(\theta) \right]^2, \quad (1)$$

where $p_i(\theta)$ is the probability density conditioned on the fixed parameter θ with measurement outcome $\{x_i\}$ for a

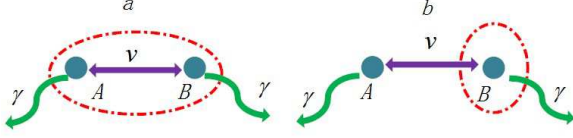


FIG. 1: Schematic diagrams of the parameter estimation adopted in this paper. Two qubits A and B couple to each other by the interaction parameter v . They also have equally local decoherent rates γ . (a) corresponds to the estimate of γ taking the two qubits as a whole, (b) corresponds to the estimate of v only using qubit B.

discrete observable X . The classical Fisher information characterizes the inverse variance of the asymptotic normality of a maximum-likelihood estimator.

Extending to quantum regime, quantum Fisher information of a parameterized quantum states $\rho(\theta)$ is defined as

$$F_\theta = \text{Tr}[\rho(\theta)L^2], \quad (2)$$

where θ is the parameter to be measured, and L is the symmetric logarithmic derivative determined by

$$\frac{d\rho(\theta)}{d\theta} = \frac{1}{2}[\rho(\theta)L + L\rho(\theta)]. \quad (3)$$

With the spectrum decomposition $\rho_\theta = \sum_k \lambda_k |k\rangle\langle k|$, its QFI with respect to θ is given as [17–19]

$$F_\theta = \sum_k \frac{(\partial_\theta \lambda_k)^2}{\lambda_k} + 2 \sum_{k,k'} \frac{(\lambda_k - \lambda_{k'})^2}{\lambda_k + \lambda_{k'}} |\langle k | \partial_\theta k' \rangle|^2. \quad (4)$$

Here $\lambda_k > 0$ and $\lambda_k + \lambda_{k'} > 0$. The first term in Eq. (4) is just the classical Fisher information Eq. (1). Then, the second term can be considered as the quantum contribution. According to quantum Cramér-Rao (QCR) inequality, the variance $\text{Var}(\hat{\theta})$ of any unbiased estimator $\hat{\theta}$ satisfies

$$\text{Var}(\hat{\theta}) \geq \frac{1}{MF_\theta}, \quad (5)$$

where M is the times of measurement. QCR embodies the ultimate limit to the precision of the estimate of θ . With a larger quantum Fisher information, the parameter θ can be estimated more accurately.

III. QUANTUM FISHER INFORMATION OF TWO QUBITS

A. Model

In order to study the the precision of parameter estimation, we consider a model consisting of two identically interacting qubits A and B [20]. This simple model can make us get analytical results for the time evolution of

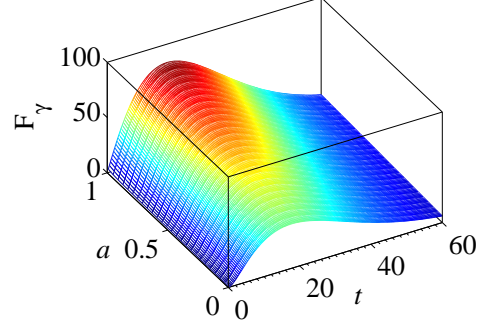


FIG. 2: The variation of F_γ in Eq. (14) with respect to the time t and the parameter a . Here the decoherence parameter $\gamma = 0.1$.

QFI. Each qubit is a two-level system with an excited state $|e\rangle$ and a ground state $|g\rangle$. The interacting Hamiltonian is given by

$$H = \frac{1}{2}v(S_A^+ S_B^- + h.c.). \quad (6)$$

Here v denotes the interaction strength between the two qubits, S_j^+ and S_j^- ($j=A, B$) are the qubit's raising and lowering operators, respectively. Furthermore, each qubit interacts independently with its Markovian environment. The decoherence may arise from the spontaneous emission of the excited state. The dynamics of the system can be treated by quantum Liouville equation ($\hbar = 1$)

$$\frac{d\rho(t)}{dt} = -i[H, \rho] - \sum_{j=A,B} \frac{\gamma_j}{2} (S_j^+ S_j^- \rho - 2S_j^- \rho S_j^+ + \rho S_j^+ S_j^-), \quad (7)$$

where $\rho(t)$ is the density operator of the two qubits, γ_j are their spontaneous decay rates. The number of elements in the density matrix $\rho(t)$ is 16.

In this paper, we consider a simple class of initial state

$$\rho_0 = \frac{1}{3}(a|ee\rangle\langle ee| + d|gg\rangle\langle gg| + |\psi\rangle\langle\psi|) = \frac{1}{3} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & z & 0 \\ 0 & z^* & 1 & 0 \\ 0 & 0 & 0 & 1-a \end{pmatrix}, \quad (8)$$

with $|\psi\rangle = |eg\rangle + z|ge\rangle$, and the parameters satisfying $0 \leq a \leq 1$, $d=1-a$ and $z = \exp(i\chi)$ [21]. This initial state is just the input state for the parameter estimation. The concurrence of ρ_0 is

$$C(\rho_0) = \frac{2}{3}[1 - \sqrt{a(1-a)}]. \quad (9)$$

It is easy to prove that the solution of the quantum Liouville equation Eq. (7) preserves the form in Eq. (8) all the time [20].

After some straightforward calculations, the nonzero

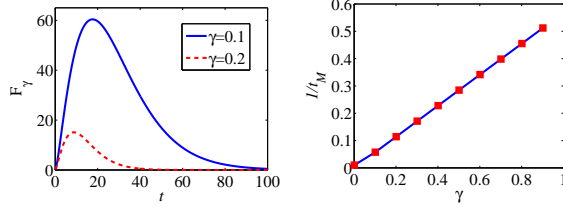


FIG. 3: Left panel: The evolution of F_γ with respect to time t under different decoherence strengths. A large decoherence rate makes the precision of estimation decrease. Right panel: The variation of $1/t_M$ in term of γ . They are linearly related. Here t_M denotes the time at which F_γ gets the maximum value. The parameter is $a = 0.5$.

elements of density matrix $\rho(t)$ are found as

$$\begin{aligned} \rho_{11} &= \frac{1}{3}ap(-t)^2, \\ \rho_{22} &= \frac{1}{3}p(-t)^2[p(t)(a - \sin(\chi) \sin(2tv) + 1) - a], \\ \rho_{23} &= \frac{1}{6}p(-t)z^*[-(-1 + z^2) \cos(2tv) + z^2 + 1], \\ \rho_{33} &= \frac{1}{3}p(-t)^2[p(t)(a + \sin(\chi) \sin(2tv) + 1) - a], \\ \rho_{32} &= \rho_{23}^*, \rho_{44} = 1 - \rho_{11} - \rho_{22} - \rho_{33}. \end{aligned} \quad (10)$$

Here we have assumed that two qubits have equal decay rates $\gamma_A = \gamma_B = \gamma$ and $p(t) \equiv e^{\gamma t}$.

After the direct diagonalization, the eigenvalues of $\rho(t)$ is given as

$$\begin{aligned} \lambda_1 &= \frac{1}{3}ap(-t)^2, \quad \lambda_2 = \frac{1}{3}ap(-t)^2(p(t) - 1), \\ \lambda_3 &= \frac{1}{3}p(-t)^2[a - 2(a + 1)p(t)] + 1, \\ \lambda_4 &= \frac{1}{3}p(-t)^2[(a + 2)p(t) - a], \end{aligned} \quad (11)$$

with the corresponding eigenvectors

$$\begin{aligned} |1\rangle &= (1, 0, 0, 0)^T, \quad |3\rangle = (0, 0, 0, 1)^T, \\ |2\rangle &= N_1(0, -2e^{i\chi}(\Delta + 1), \Gamma, 0)^T, \\ |4\rangle &= N_1(0, -2e^{i\chi}(\Delta - 1), \Gamma, 0)^T. \end{aligned} \quad (12)$$

Here

$$\Gamma = 2e^{2i\chi} \sin^2(tv) + \cos(2tv) + 1, \Delta = \sin(\chi) \sin(2tv), \quad (13)$$

N_1 is a normalized constant and A^T denotes the transposition of the matrix A .

B. Estimation on decoherent rates

As mentioned above, in this paper we have assumed that the decoherent rates of the two qubits A and B are equal $\gamma_A = \gamma_B = \gamma$. If one considers this decoherence as a quantum channel, it is reasonable to adopt the two qubits as a whole to estimate the decoherent parameter γ of the environment, as shown in Fig. 1(a). In order to achieve this goal, we adopt the upper mentioned quantum estimation method and focus on the the dynamics of quantum Fisher information. According to Eq. (4), the quantum Fisher information of $\rho(t)$ with respect to γ is

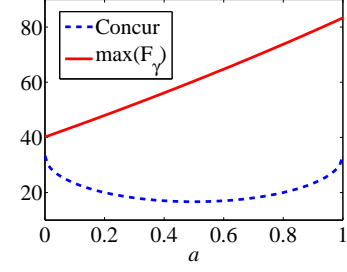


FIG. 4: The variation of the maximum value of $F_\gamma(t)$ with respect to a . The curve of concurrence has been amplified by 50 times for clarity. The parameter is $\gamma = 0.1$.

expressed as

$$F_\gamma(t) = \frac{t^2}{3} \left[\frac{4(a^2 - a + 1)}{-2ap(t) + a - 2p(t) + 3p(t)^2} + \frac{(a+2)^2}{ap(t) - a + 2p(t)} + \frac{a}{p(t) - 1} \right]. \quad (14)$$

It's obvious that $F_\gamma(t)$ is independent of parameters v and χ . This can be easily understood by the following fact: only the decay process from the excited state $|e\rangle$ to the ground state $|g\rangle$ contributes to $F_\gamma(t)$. This process is independent of their interaction and initial coherence $\exp(i\chi)$. Note also that the eigenvectors $|k\rangle$ are independent of γ , so $F_\gamma(t)$ only has a classical part according to Eq. (4). Our results indicate that from the perspective of quantum Fisher information, this entangled mixed state $\rho(t)$ does not show any quantum coherence. The relationship between quantum Fisher information and entanglement will be studied in future.

In addition, it would be interesting to consider the asymptotic limits of $F_\gamma(t)$. On the one hand, in the limit $\gamma \rightarrow 0$, that the system approaches the unitary evolution,

$$F_\gamma(t) \propto \frac{t^2}{\exp(\gamma t) - 1}. \quad (15)$$

In this case, the state preserves its phase coherence and QFI is an increase function at the initial time evolution. On the other hand,

$$F_\gamma(t) \propto \exp(-2\gamma t) \quad (16)$$

in the completely decoherent limit $\gamma \rightarrow \infty$. In this case, the state loses its coherence and QFI is exponential decay. As $\frac{t^2}{\exp(\gamma t) - 1}$ is an increasing function of time and $\exp(-2\gamma t)$ is a decreasing one, F_γ must have a maximum at a finite time.

We also adopt the numerical simulations to display other behaviors of quantum Fisher information. In Fig. 2, we show the evolution of quantum Fisher information with respect to time t and parameter a in Eq. (8). It is clear that $F_\gamma(t)$ has a maximum value at a finite time, and approaches zero in the long time limit. The left panel in Fig. 3 shows the effect of the decoherent parameter γ on quantum Fisher information. F_γ drops considerably with a small increment of γ . The decrease of the maximum value of QFI reflects that the parameter estimation of the open system becomes more inaccurate.

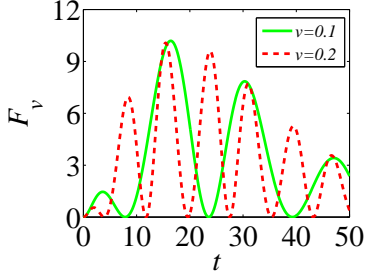


FIG. 5: The effect of interacting parameter v on the variation of F_v with respect to time t . $F_v(t)$ shows oscillation behavior. Other parameters are $\gamma = 0.1$, $a = 0.8$ and $\chi = 0.5$.

The similar result has been obtained in Ref. [14]. Moreover, we also found that $1/t_M$ is linearly related to γ , i.e., $\gamma t_M = \text{Const.}$, as shown in the right panel of Fig. 3. Here t_M denotes the time at which F_γ gets the maximum value.

As the maximum value of QFI implies the largest precision to estimate γ , in Fig. 4 we plot the maximum value of QFI $\max(F_\gamma)$ in terms of a . It shows that $\max(F_\gamma)$ monotonously increases with respect to a . This relationship can be understood from the following arguments. The information of γ comes from the spontaneous emission of the excited state in the initial state. For $a = 1$, $\rho_0 = \frac{1}{3}(|ee\rangle\langle ee| + |\psi\rangle\langle\psi|)$, which has the largest probability in the excited state compared to the other value of a . So this state corresponds to the largest value of $\max(F_\gamma)$. With the similar reason, $\rho_0 = \frac{1}{3}(|gg\rangle\langle gg| + |\psi\rangle\langle\psi|)$ with $a = 0$ has the smallest value of $\max(F_\gamma)$.

More interestingly, the input state at $a = 0$ has the maximum entanglement, which corresponds to the minimum value of $\max(F_\gamma)$. Generally speaking, entanglement is expected to be helpful in estimating the parameters of a quantum channel [22, 23]. However, its belief is not always true. For example, a two-qubit maximally entangled state only achieves the best precision estimation in some *limited* range of depolarization [24–27]. Based on the dynamics of quantum Fisher information, we also check that parameter estimation is not enhanced by the entanglement.

C. Estimation on interacting strength

The two qubits A and B interacts with each other. Considering the exchanging symmetry between them, it's natural that one chooses one qubit such as qubit B to estimate the coupling strength v , as shown in Fig. 1(b). Its reduced density matrix $\rho_B = \text{Tr}_A(\rho)$

$$\rho_B = \begin{pmatrix} \frac{1}{3}p(-t)\Omega(t) & 0 \\ 0 & 1 - \frac{1}{3}p(-t)\Omega(t) \end{pmatrix}. \quad (17)$$

Adopting a similar procedure as above, the QFI of ρ_B

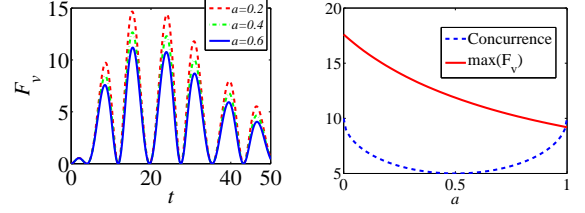


FIG. 6: Left panel: The evolution of $F_v(t)$. Different value of a only affect the amplitude of $F_v(t)$. Right panel: The variation of the maximum value of $F_v(t)$ with respect to a . The curve of concurrence has been amplified by 15 times for clarity. Other parameters are $\gamma = 0.1$, $v = 0.2$, $\chi = 0.5$

with respect to the coupling strength v is

$$F_v(t) = \frac{[2t \sin(\chi) \cos(2vt)]^2}{\Omega(t)[3p(t) - \Omega(t)]} \quad (18)$$

with $\Omega(t) = 1 + a + \sin(2vt) \sin(\chi)$. In contrast to the two qubits Fisher information Eq. (14), here the initial coherence between A and B plays a role. For example, $F_v(t) = 0$ if $\chi=0$ and π , while $F_v(t) > 0$ for the other values of χ .

Eq. (18) also displays that the information of the interaction parameter v is embedded in the evolution of $F_v(t)$. The variation of $F_v(t)$ in terms of time is shown in Fig. 5. It displays the oscillating behavior and has multi-peaks, which is distinct from the single-peak of $F_\gamma(t)$ as shown in Fig. (3).

We also study the effect of the initial parameter a on quantum Fisher information $F_v(t)$, as an example shown in Fig. 6. With different values of a , the amplitudes of $F_v(t)$ are changed. The maximum value of $F_v(t)$ is a decreasing function of a . Compared to Fig. 4, in this figure the input state at $a = 1$ has the maximum entanglement, but it corresponds to the minimum value of $\max(F_\gamma)$.

IV. CONCLUSION

In conclusion, we have analyzed the dynamics of quantum Fisher information, which is related to Cramér-Rao inequality in quantum estimation theory, for two initially entangled qubits under decoherence. We have observed that for a special class of X -state, its quantum Fisher information F_γ with respect to the decoherent parameter only has the classical part. Moreover, the dynamical evolution of quantum Fisher information shows $\frac{t^2}{\exp(\gamma t) - 1}$ and $\exp(-2\gamma t)$ in the unitary and completely decoherent limit, respectively. We also study the estimation of the coupling strength between two qubits by the dynamics of quantum Fisher information $F_v(t)$ for a reduced qubit. It is oscillation in term of time, which implies the information of the interaction parameter may be obtained by the Fourier transformation of $F_v(t)$. In both cases, we do not observe that quantum Fisher information is enhanced by the entanglement of the input state. The relationship

between quantum Fisher information and quantum entanglement deserves further study.

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